

Mathematical Analysis I

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Preface

The recent European Programme Specifications have forced a reassessment of the structure and syllabi of the entire system of Italian higher education, and an ensuing rethinking of the teaching material.

Nowadays many lecture courses, especially rudimentary ones, demand that students master a large amount of theoretical and practical knowledge in a span of just few weeks, in order to gain a small number of credits. As a result, instructors face the dilemma of how to present the subject matter. They must make appropriate choices about lecture content, the comprehension level required from the recipients, and which kind of language to use.

This textbook is meant to help students acquire the basics of Calculus in curricula where mathematical tools play a crucial part (so Engineering, Physics, Computer Science and the like). The fundamental concepts and methods of Differential and Integral Calculus for functions of one real variable are presented with the primary purpose of letting students assimilate their effective employment, but with critical awareness. The general philosophy inspiring our approach has been to simplify the system of notions available prior to the university reform; at the same time we wished to maintain the rigorous exposition and avoid the trap of compiling a mere formulary of ready-to-use prescriptions.

From this point of view, the treatise is ‘stratified’ in three layers, each corresponding to increasingly deeper engagement by the user. The intermediate level corresponds to the unabridged text. Notions are first presented in a naïve manner, and only later defined precisely. Their features are discussed, and computational techniques related to them are exhaustively explained. Besides this, the fundamental theorems and properties are followed by proofs, which are easily recognisable by the font’s colour.

At the elementary level the proofs and the various remarks should be skipped. For the reader’s sake, essential formulas, and also those judged important, have been highlighted in blue, and gray, respectively. Some tables, placed both throughout and at the end of the book, collect the most useful formulas. It was not our desire to create a hierarchy-of-sorts for theorems, instead to leave the instructor free to make up his or her own mind in this respect.

The deepest-reaching level relates to an internet website, and enables the strongly motivated reader to explore further into the subject. We believe that the general objectives of the Programme Specifications are in line with the fact that willing and able pupils will build a solid knowledge, in the tradition of Italy's academic education. The book contains several links to a webpage where the reader will find complements to, and insight in various topics. In this fashion every result that is stated possesses a corresponding proof.

To make the approach to the subject less harsh, and all the more gratifying, we have chosen a casual presentation in the first two chapters, where relevant definitions and properties are typically part of the text. From the third chapter onwards they are highlighted by the layout more discernibly. Some definitions and theorems are intentionally not stated in the most general form, so to privilege a brisk understanding. For this reason a wealth of examples are routinely added along the way right after statements, and the same is true for computational techniques. Several remarks enhance the presentation by underlining, in particular, special cases and exceptions. Each chapter ends with a large number of exercises that allow one to test on the spot how solid one's knowledge is. Exercises are grouped according to the chapter's major themes and presented in increasing order of difficulty. All problems are solved, and at least half of them chaperone the reader to the solution.

We have adopted the following graphical conventions for the constituent building blocks: definitions appear on a gray background, theorems' statements on blue, a vertical coloured line marks examples, and boxed exercises, like 12., indicate that the complete solution is provided.

An Italian version of this book has circulated for a number of years in Italy, and has been extensively tested at the Politecnico in Turin. We wish to dedicate this volume to Professor Guido Weiss of Washington University in St. Louis, a master in the art of teaching. Generations of students worldwide have benefited from Guido's own work as a mathematician; we hope that his own clarity is at least partly reflected in this textbook.

We are thankful to the many colleagues and students whose advice, suggestions and observations have allowed us to improve the exposition. Special thanks are due to Dr. Simon Chiossi, for the careful and effective work of translation.

Online additional material

At the address

http://calvino.polito.it/canuto-tabacco/analysis_1

further material and complementary theory are available. These online notes are organised under the following headings:

- Principle of Mathematical Induction
- The number e
- Elementary functions
- Limits
- Continuous functions
- Sequences
- Numerical series
- Derivatives
- De l'Hôpital's Theorem
- Convex functions
- Taylor expansions
- Cauchy integral
- Riemann integral
- Improper integrals

and provides, in particular, rigorous proofs for the statements that are not shown in the text. The reader is encouraged to refer to the relevant section whenever the symbol \rightsquigarrow appears throughout the treatise, as in \rightsquigarrow **The number e** .

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